

Multiresolution Analysis Based on Coalescence Hidden-variable FIF

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Abstract

In the present paper, multiresolution analysis arising from Coalescence Hidden-variable Fractal Interpolation Functions (CHFIFs) is accomplished. The availability of a larger set of free variables and constrained variables with CHFIF in multiresolution analysis based on CHFIFs provides more control in reconstruction of functions in $L_2(\mathbb{R})$ than that provided by multiresolution analysis based only on Affine Fractal Interpolation Functions (AFIFs). In our approach, the vector space of CHFIFs is introduced, its dimension is determined and Riesz bases of vector subspaces $\mathbb{V}_k, k \in \mathbb{Z}$, consisting of certain CHFIFs in $L_2(\mathbb{R}) \cap C_0(\mathbb{R})$ are constructed. As a special case, for the vector space of CHFIFs of dimension 4, orthogonal bases for the vector subspaces $\mathbb{V}_k, k \in \mathbb{Z}$, are explicitly constructed and, using these bases, compactly supported continuous orthonormal wavelets are generated.

Keywords: Fractal, Interpolation, Iteration, Affine, Coalescence, Attractor, Multiresolution Analysis, Riesz Basis, Orthogonal Basis, Scaling Function, Wavelets

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1 Introduction

The theory of multiresolution analysis provides a powerful method to construct wavelets having far reaching applications in analyzing signals and images [11, 14]. They permit efficient representation of functions at multiple levels of detail, i.e. a function $f \in L_2(\mathbb{R})$, the space of real valued functions g satisfying $\|g\|_{L^2} = \int_{\mathbb{R}} |g(x)|^2 dx < \infty$, could be written as limit of successive approximations, each of which is smoothed version of f . The multiresolution analysis was first introduced by Mallat [10] and Meyer [13] using a single function. The multiresolution analysis based upon several functions was developed in [6, 7, 9]. In [8], multiresolution analysis of $L^2(\mathbb{R})$ were generated from certain classes of Affine Fractal Interpolation Functions (AFIFs). Such results were then generalized to several dimensions in [3] and [4]. In [5], orthonormal basis for the vector space of AFIFs were explicitly constructed. A few years later, Donovan et al [2] constructed orthogonal compactly supported continuous wavelets using multiresolution analysis arising from AFIFs. The interrelations among AFIFs, Multiresolution Analysis and Wavelets are treated in detail by Massopust [12]. However, multiresolution analysis of $L_2(\mathbb{R})$ based on Coalescence Hidden-variable Fractal Interpolation Functions (CHFIFs) which exhibits both self-affine and non-self-affine nature has hitherto remained unexplored. In the present work, such a multiresolution analysis is accomplished. The availability of a larger set of free variables and constrained variables in multiresolution analysis based on CHFIFs provides more control in reconstruction of functions in $L_2(\mathbb{R})$ than that provided by multiresolution analysis based only on affine FIFs. Further, orthogonal bases consisting of dilations and translations of scaling functions, for the vector subspaces $\mathbb{V}_k, k \in \mathbb{Z}$, are explicitly constructed and, using these bases, compactly supported continuous orthonormal wavelets are generated in the present work.

The organization of the paper is as follows: In Section 2, a brief introduction on the construction of CHFIF is given, the vector space of CHFIFs is introduced and a few auxiliary results, including a result on determination of dimension of this vector space, are found. In Section 3, Riesz bases of vector subspaces $\mathbb{V}_k, k \in \mathbb{Z}$, consisting of certain CHFIFs in $L_2(\mathbb{R}) \cap C_0(\mathbb{R})$ are constructed. The multiresolution analysis of $L_2(\mathbb{R})$ is then carried out in terms of nested sequences of vector subspaces $\mathbb{V}_k, k \in \mathbb{Z}$. As a special case, for the vector space of CHFIFs of dimension 4, orthogonal bases for the vector subspaces $\mathbb{V}_k, k \in \mathbb{Z}$, are explicitly constructed in Section 4 and, using these bases, compactly supported continuous orthonormal wavelets are generated in the same section.

2 Preliminaries and Auxiliary Results

In this section, first a brief introduction on the construction of CHFIF is given. This is followed by the development of some auxiliary results needed for the multiresolution analysis generated by CHFIFs.

A Coalescence Hidden-variable Fractal Interpolation Function (CHFIF) is constructed such that the graph of CHFIF is attractor of an IFS. Let the interpolation data be $\{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1, \dots, N\}$, where $-\infty < x_0 < x_1 < \dots < x_N < \infty$. By introducing a set of real parameters z_i for $i = 0, 1, \dots, N$, consider the generalized interpolation data $\{(x_i, y_i, z_i) \in \mathbb{R}^3 : i = 0, 1, \dots, N\}$. The contractive homeomorphisms $L_n : I \rightarrow I_n$ for $n = 1, \dots, N$, are defined by

$$L_n(x) = a_n x + b_n \quad (2.1)$$

where, $a_n = \frac{x_n - x_{n-1}}{x_N - x_0}$ and $b_n = \frac{x_N x_{n-1} - x_0 x_n}{x_N - x_0}$. For $n = 1, \dots, N$, define the maps $\omega_n : I \times \mathbb{R}^2 \rightarrow I \times \mathbb{R}^2$ by

$$\omega_n(x, y, z) = (L_n(x), F_n(x, y, z)) \quad (2.2)$$

where, the functions $F_n : I \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by,

$$F_n(x, y, z) = (\alpha_n y + \beta_n z + p_n(x), \gamma_n z + q_n(x)) \quad (2.3)$$

satisfy the join-up conditions

$$F_n(x_0, y_0, z_0) = (y_{n-1}, z_{n-1}) \quad \text{and} \quad F_n(x_N, y_N, z_N) = (y_n, z_n). \quad (2.4)$$

Here, the variables α_n, γ_n are free variables and β_n are constrained variables such that $|\alpha_n| < 1, |\gamma_n| < 1$ and $|\beta_n| + |\gamma_n| < 1$ and the functions $p_n(x)$ and $q_n(x)$ are linear polynomials given by

$$p_n(x) = c_n x + d_n \quad \text{and} \quad q_n(x) = e_n x + h_n. \quad (2.5)$$

It is proved in [1] that there exist a metric equivalent to Euclidean metric such that the functions ω_n , defined by $\omega_n(x, y, z) = (L_n(x), F_n(x, y, z))$, are contraction maps and, consequently,

$$\{I \times \mathbb{R}^2; \omega_n, n = 1, 2, \dots, N\} \quad (2.6)$$

is the desired IFS for construction of CHFIF. Hence, there exists an attractor A in $H(I \times \mathbb{R}^2)$ such that $A = \bigcup_{n=1}^N \omega_n(A) = \bigcup_{n=1}^N \{\omega_n(x, y, z) : (x, y, z) \in A\}$ and is graph of a continuous function $f : I \rightarrow \mathbb{R}^2$ such that $f(x_i) = (y_i, z_i)$ for $i = 0, 1, \dots, N$, i.e. $A = \{(x, f(x)) : x \in I, f(x) = (y(x), z(x))\}$. By expressing f component-wise as $f = (f_1, f_2)$, *Coalescence Hidden-variable Fractal Interpolation Function* (CHFIF) [1] is defined as the continuous function f_1 for the given interpolation data $\{(x_i, y_i) : i = 0, 1, \dots, N\}$.

In order to develop the multiresolution analysis of $L^2(\mathbb{R})$, based on CHFIF, the space of CHFIF needs to be introduced. For this purpose, let

$$t_n = (p_n, q_n) \quad (2.7)$$

and $t = (t_1, t_2, \dots, t_N)$, where p_n and q_n are linear polynomials given by (2.5). Then, $T = \{t = (t_1, \dots, t_N) : t_i = (p_i, q_i), p_i, q_i \in \mathcal{P}_1, i = 1, 2, \dots, N\}$, with usual point-wise addition and scalar multiplication, is a vector space, where \mathcal{P}_1 is the class of linear polynomials. It is easily seen that on $\mathbb{B}(I, \mathbb{R}^2)$, the set of bounded functions from I to \mathbb{R}^2 with respect to maximum metric $d^*(f, g) = \max_{x \in I} \{|f_1(x) - g_1(x)|, |f_2(x) - g_2(x)|\}$, the function Φ_t defined by

$$(\Phi_t f)(x) = F_n(L_n^{-1}(x), f(L_n^{-1}(x))) \quad (2.8)$$

for $x \in I_n = [x_{n-1}, x_n]$, $n = 1, 2, \dots, N$, is a contraction map. Therefore, by Banach contraction mapping theorem, Φ_t has a unique fixed point $f_t \in \mathbb{B}(I, \mathbb{R}^2)$. By join-up conditions (2.4), it follows that $f_t \in \mathbb{C}(I, \mathbb{R}^2)$, the set of continuous functions from I to \mathbb{R}^2 . The following proposition gives the existence of a linear isomorphism between the vector space T and the vector space $\mathbb{C}(I, \mathbb{R}^2)$.

Proposition 2.1. *The mapping $\Theta : T \rightarrow \mathbb{C}(I, \mathbb{R}^2)$ defined by $\Theta(t) = f_t$ is a linear isomorphism.*

Proof. The assertion of the proposition is proved by establishing

(i) $(af_t + f_s)_i(x) = af_{t,i}(x) + f_{s,i}(x)$, $i = 1, 2$, where f_t and $af_t + f_s$ are written component-wise as $f_t = (f_{t,1}, f_{t,2})$ and $af_t + f_s = ((af_t + f_s)_1, (af_t + f_s)_2)$, (ii) $(af_t + f_s) = f_{at+s}$, (iii) Θ is onto and (iv) Θ is one-one.

The identity (i) follows by equating the components of left and right hand side in the identity $(af_t + f_s)(x) = af_t(x) + f_s(x)$.

(ii) $(af_t + f_s) = f_{at+s}$: By the definition of Φ_t ,

$$\begin{aligned}
& (\Phi_{at+s}(af_t + f_s))(x) \\
&= F_n(L_n^{-1}(x), (af_t + f_s)(L_n^{-1}(x))) \\
&= \left(\alpha_n(af_t + f_s)_1(L_n^{-1}(x)) + \beta_n(af_t + f_s)_2(L_n^{-1}(x)) + (a p_n + \hat{p}_n)(L_n^{-1}(x)), \right. \\
&\quad \left. \gamma_n(af_t + f_s)_2(L_n^{-1}(x)) + (a q_n + \hat{q}_n)(L_n^{-1}(x)) \right)
\end{aligned}$$

Using identity (i), it follows that,

$$\begin{aligned}
& (\Phi_{at+s}(af_t + f_s))(x) \\
&= \left(\alpha_n(af_{t,1}(L_n^{-1}(x)) + f_{s,1}(L_n^{-1}(x))) + \beta_n(af_{t,2}(L_n^{-1}(x)) + f_{s,2}(L_n^{-1}(x))) \right. \\
&\quad \left. + (a p_n(L_n^{-1}(x)) + \hat{p}_n(L_n^{-1}(x))), \right. \\
&\quad \left. \gamma_n(af_{t,2}(L_n^{-1}(x)) + f_{s,2}(L_n^{-1}(x))) + (a q_n(L_n^{-1}(x)) + \hat{q}_n(L_n^{-1}(x))) \right)
\end{aligned}$$

The above equation gives the following on simplification:

$$\begin{aligned}
& (\Phi_{at+s}(af_t + f_s))(x) \\
&= a \left(\alpha_n f_{t,1}(L_n^{-1}(x)) + \beta_n f_{t,2}(L_n^{-1}(x)) + p_n(L_n^{-1}(x)), \gamma_n f_{t,2}(L_n^{-1}(x)) + q_n(L_n^{-1}(x)) \right) \\
&\quad + \left(\alpha_n f_{s,1}(L_n^{-1}(x)) + \beta_n f_{s,2}(L_n^{-1}(x)) + \hat{p}_n(L_n^{-1}(x)), \gamma_n f_{s,2}(L_n^{-1}(x)) + \hat{q}_n(L_n^{-1}(x)) \right) \\
&= af_t(x) + f_s(x)
\end{aligned}$$

Therefore, $af_t + f_s$ is a fixed point of Φ_{at+s} for all $a \in \mathbb{R}$ and $t, s \in T$. By uniqueness of fixed point of Φ_{at+s} , it follows that $(af_t + f_s) = f_{at+s}$.

(iii) Θ is onto : Let $f = (f_1, f_2) \in \mathbb{C}(I, \mathbb{R}^2)$. Define $q_i(f) = f_2 \circ L_i - \gamma_i f_2$ and $p_i(f) = f_1 \circ L_i - \alpha_i f_1 - \beta_i f_2$ for $i = 1, \dots, N$. Suppose $t(f) = (t_1(f), t_2(f), \dots, t_N(f))$, where $t_i(f) = (p_i(f), q_i(f))$. Then $t(f) \in T$ whenever $f \in \mathbb{C}(I, \mathbb{R}^2)$. Also $f_{t(f)} = f$.

(iv) Θ is one-one : Let $f_t(x) = (0, 0)$ for all values of $x \in I$. Then,

$$\begin{aligned}
f_t(x) = (0, 0) &\Leftrightarrow \Phi_t(f_t)(x) = (0, 0) \\
&\Leftrightarrow F_n(L_n^{-1}(x), f_t(L_n^{-1}(x))) = (0, 0) \\
&\Leftrightarrow (p_n, q_n) = (0, 0) \text{ for every } n \\
&\Leftrightarrow t = (0, \dots, 0)
\end{aligned}$$

□

To introduce the space of CHFIFs, let the set \mathcal{S}_0 consisting of functions $f : I \rightarrow \mathbb{R}^2$ be defined as $\mathcal{S}_0 = \{f : f = (f_1, f_2), f_1 \text{ is a CHFIF passing through } \{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1, \dots, N\} \text{ and } f_2 \text{ is an AFIF passing through } \{(x_i, z_i) \in \mathbb{R}^2 : i = 0, 1, \dots, N\}\}$. Then, \mathcal{S}_0 is a vector space, with usual point-wise addition and scalar multiplication. The space of CHFIFs is now defined as follows:

Definition 2.1. Let \mathcal{S}_0^1 be the set of functions $f_1 : I \rightarrow \mathbb{R}$ that are first components of functions $f \in \mathcal{S}_0$. The **space of CHFIFs** is the set \mathcal{S}_0^1 together with the maximum metric $d^*(f, g) = \max_{x \in I} |f(x) - g(x)|$.

It is easily seen that the space of CHFIFs \mathcal{S}_0^1 is also a vector space with point-wise addition and scalar multiplication. The following proposition gives the dimension of \mathcal{S}_0^1 :

Proposition 2.2. *The dimension of space of CHFIFs is $2N$.*

Proof. Consider the operator $\Phi_t f = (\Phi_{t,1} f_1, \Phi_{t,2} f_2)$. The operators $\Phi_{t,i} : \mathbb{B}(I, \mathbb{R}) \rightarrow \mathbb{B}(I, \mathbb{R})$, $i = 1, 2$, where $\mathbb{B}(I, \mathbb{R})$ is the set of bounded functions, satisfy

$$\Phi_{t,1} f_1(x) = \alpha_n f_1(L_n^{-1}(x)) + \beta_n f_2(L_n^{-1}(x)) + p_n(L_n^{-1}(x)) \quad (2.9)$$

and

$$\Phi_{t,2} f_2(x) = \gamma_n f_2(L_n^{-1}(x)) + q_n(L_n^{-1}(x)) \quad (2.10)$$

for $x \in [x_{n-1}, x_n]$. By Proposition 2.1 and (2.10), it follows that f_2 is completely determined by $f_2(\frac{i}{N})$ for $i = 0, 1, \dots, N$. Further, it follows by (2.9) that f_1 depends on f_2 . Then, for $f = (f_1, f_2) \in \mathcal{S}_0$, the function f_1 is the unique CHFIF passing through $(\frac{i}{N}, y_i)$, while the function f_2 is the unique AFIF passing through $(\frac{i}{N}, z_i)$. Hence,

$$\text{dimension of } \mathcal{S}_0 = 2(N + 1). \quad (2.11)$$

Consider the projection map $P : \mathcal{S}_0 \rightarrow \mathcal{S}_0^1$. Then, Kernel of $P = \{f \in \mathcal{S}_0 \text{ such that } P(f) = 0\}$ is a proper subset of \mathcal{S}_0 and consists of elements in the form $(0, 0)$ and $(0, f_2)$. For the element $(0, f_2) \in \text{Ker } P$, it is observed that $\beta_n f_2(L_n^{-1}(x)) + p_n(L_n^{-1}(x)) = 0$ for $x \in I_n$. Hence, for all $x \in I$, it is seen that $f_2(x) = \frac{-1}{\beta_n} p_n(x)$. With $x = x_0$, it follows that $c_i = \frac{\beta_i}{\beta_1} c_1$ and $d_i = \frac{\beta_i}{\beta_1} d_1, i = 2, \dots, N$. Consequently, if $(0, f_2) \in \text{Ker } P$ then f_2 is a linear polynomial. So, dimension of $\text{Ker } P = 2$. Therefore, by Rank-Nullity Theorem [56], dimension of $\mathcal{S}_0^1 = \text{dimension of } \mathcal{S}_0 - \text{dimension of Ker } P = 2(N + 1) - 2 = 2N$. \square

Remark 2.2. By Proposition 2.1 and (2.11), it follows that the map $\theta : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \rightarrow \mathcal{S}_0$ defined by $\theta(y, z) = f$ is a linear isomorphism, where $f = (f_1, f_2) \in \mathcal{S}_0$, f_1 is the unique CHFIF passing through the points (x_i, y_i) and f_2 is the unique AFIF passing through the points (x_i, z_i) , $y = (y_0, y_1, \dots, y_N)$ and $z = (z_0, z_1, \dots, z_N)$. Thus, \mathcal{S}_0 is linearly isomorphic to $\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$. Consider the metric space $(\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}, d_{\mathbb{R}^{2(N+1)}})$, where $d_{\mathbb{R}^{2(N+1)}}$ is given by $d_{\mathbb{R}^{2(N+1)}}(y \times z, \bar{y} \times \bar{z}) = \max_{0 \leq i \leq N} (|y_i - \bar{y}_i|, |z_i - \bar{z}_i|)$, $y = (y_0, y_1, \dots, y_N)$, $z = (z_0, z_1, \dots, z_N)$, $\bar{y} = (\bar{y}_0, \bar{y}_1, \dots, \bar{y}_N)$ and $\bar{z} = (\bar{z}_0, \bar{z}_1, \dots, \bar{z}_N)$. Then, with the metric d^* on the set \mathcal{S}_0 , it is observed by (2.8) that the maps θ and θ^{-1} are continuous. Hence \mathcal{S}_0 is closed and complete subspace of $L_2(\mathbb{R})$.

Remark 2.3. Let $\{f_{n,1}\}$ be a sequence in \mathcal{S}_0^1 such that $\lim_{n \rightarrow \infty} f_{n,1} = f_1^*$ and $\{f_n = (f_{n,1}, f_{n,2})\}$ be a convergent sequence in \mathcal{S}_0 , where $f_{n,2}$ are AFIFs. Since \mathcal{S}_0 is closed, $\lim_{n \rightarrow \infty} f_n = f^* \equiv (f_1^*, f_2^*) \in \mathcal{S}_0$. Thus, $f_1^* \in \mathcal{S}_0^1$ and consequently, \mathcal{S}_0^1 is closed subspace of $L_2(\mathbb{R})$.

3 Multiresolution Analysis Based on CHFIF

In this section, the multiresolution analysis of $L_2(\mathbb{R})$ is generated by using a finite set of CHFIFs. For this purpose, the sets $\mathbb{V}_k, k \in \mathbb{Z}$, consisting of collection of CHFIFs are defined. It is first shown that the sets \mathbb{V}_k form a nested sequence. The multiresolution analysis of $L_2(\mathbb{R})$ is then generated by constructing Riesz bases of vector subspaces \mathbb{V}_k consisting of orthogonal functions in $L_2(\mathbb{R})$.

To introduce certain sets of CHFIFs needed for multiresolution of $L_2(\mathbb{R})$, let $L_2(\mathbb{R}, \mathbb{R}^2)$ be a collection of functions $f : \mathbb{R} \rightarrow \mathbb{R}^2$ such that $f = (f_1, f_2)$ and $f_1, f_2 \in L_2(\mathbb{R})$ and $C_0(\mathbb{R}, \mathbb{R}^2)$ be a collection of functions $f : \mathbb{R} \rightarrow \mathbb{R}^2$ such that $f = (f_1, f_2)$ and $f_1, f_2 \in C_0(\mathbb{R})$, the set of all real valued continuous functions defined on \mathbb{R} which vanish at infinity. Define the set $\tilde{\mathbb{V}}_0$ as

$$\tilde{\mathbb{V}}_0 = \tilde{S}_0 \bigcap L_2(\mathbb{R}, \mathbb{R}^2) \bigcap C_0(\mathbb{R}, \mathbb{R}^2)$$

where, $\tilde{S}_0 = \{f : f = (f_1, f_2), f_1|_{[i-1, i)} \text{ is a CHFIF and } f_2|_{[i-1, i)} \text{ is an AFIF, } i \in \mathbb{Z}\}$.

That the set $\tilde{\mathbb{V}}_0$ is not empty is easily seen by considering a function $f = (f_1, f_2) \in \mathcal{S}_0$, with $f(x_0) = (0, 0) = f(x_N)$ and $f(x) = (0, 0)$ for $x \notin I$, which obviously belongs to $\tilde{\mathbb{V}}_0$. Let, for $k \in \mathbb{Z}$,

$$\tilde{\mathbb{V}}_k = \{f : f(N^{-k} \cdot) \in \tilde{\mathbb{V}}_0\}.$$

The sets $\tilde{\mathbb{V}}_0$ and $\tilde{\mathbb{V}}_k$ are easily seen to be closed sets as follows: Let $\{f_n\}$ be a sequence in $\tilde{\mathbb{V}}_0$ such that $\lim_{n \rightarrow \infty} f_n = f^* = (f_1^*, f_2^*)$. Now, $\lim_{n \rightarrow \infty} f_n|_{[i-1, i)} = f^*|_{[i-1, i)} = (f_1^*|_{[i-1, i)}, f_2^*|_{[i-1, i)})$. By Remark 2.2 and Remark 2.3, it is observed that $f_1^*|_{[i-1, i)}$ is a CHFIF and $f_2^*|_{[i-1, i)}$ is an AFIF, $i \in \mathbb{Z}$. Thus, $f^* \in \tilde{S}_0$, which implies that \tilde{S}_0 is a closed set. Consequently, $\tilde{\mathbb{V}}_0$ and $\tilde{\mathbb{V}}_k, k \in \mathbb{Z}$ are closed sets. Now, consider sets \mathbb{V}_0 and $\mathbb{V}_k, k \in \mathbb{Z} \setminus 0$, of CHFIFs defined as follows:

$$\mathbb{V}_0 = \{f_1 : f_1 \text{ is the first component of some } f = (f_1, f_2) \in \tilde{\mathbb{V}}_0\} \quad (3.1)$$

and

$$\mathbb{V}_k = \{f_1 : f_1(N^{-k} \cdot) \in \mathbb{V}_0\}. \quad (3.2)$$

The sets $\mathbb{V}_k, k \in \mathbb{Z}$, with L_2 -norm, are subspaces of $L_2(\mathbb{R})$. The following proposition shows that the subspaces $\mathbb{V}_k, k \in \mathbb{Z}$, of $L_2(\mathbb{R})$, form a nested sequence:

Proposition 3.1. *The subspaces $\mathbb{V}_k, k \in \mathbb{Z}$ are vector subspaces of $L_2(\mathbb{R})$ and form a nested sequence $\dots \supseteq \mathbb{V}_{-1} \supseteq \mathbb{V}_0 \supseteq \mathbb{V}_1 \supseteq \dots$*

Proof. It follows from Proposition 2.1 that the sets $\mathbb{V}_k, k \in \mathbb{Z}$ are vector subspaces of $L_2(\mathbb{R})$. Now, to show $\mathbb{V}_k \supseteq \mathbb{V}_{k+1}$ for all $k \in \mathbb{Z}$, it suffices to prove the inclusion relation for $k = 0$. Let $f \in \mathbb{V}_1$. Then, $f|_{[0, N)} = g_1|_{[0, N)}$ for some $g = (g_1, g_2) \in \tilde{\mathbb{V}}_1$. If $G = \text{graph}(g|_{[0, N)})$ then, $G = \bigcup_{i=1}^N w_i(G)$ implies, for $j \in \{1, \dots, N\}$, $w_j(G) = \bigcup_{i=1}^N w_j \circ w_i \circ w_j^{-1}(w_j(G))$, where $w_i(G) = (L_i(x), F_i(x, y, z))$ for all $(x, y, z) \in G$, $i = 1, \dots, N$. Expressing w_i and $w_j \circ w_i \circ w_j^{-1}$ in matrix form as $w_i(x, y, z) = A_i(x, y, z) + B_i$ and $w_j \circ w_i \circ w_j^{-1}(x, y, z) = A_{i,j}(x, y, z) + B_{i,j}$, it is observed that non-zero entries in matrices A_i and $A_{i,j}$ occur at the same places. Consequently, $w_j(G)$ is graph of $g|_{[j-1, j)}$, so that $g \in \tilde{\mathbb{V}}_0$. It therefore follows that $g_1|_{[j-1, j)}$ is a CHFIF on the interval $[j-1, j)$. Thus, the function $f|_{[j-1, j)} = g_1|_{[j-1, j)}$ is a CHFIF on the interval $[j-1, j)$ and consequently, $f \in \mathbb{V}_0$. \square

In order to generate a multiresolution analysis of $L_2(\mathbb{R})$ using CHFIFs, the inner product on the space $\mathbb{V}_k, k \in \mathbb{Z}$ is defined by $\langle f_1, \hat{f}_1 \rangle = \int_{\mathbb{R}} f_1(x) \hat{f}_1(x) dx$. Using the following recurrence relations,

$$f_1(L_n(x)) = \alpha_n f_1(x) + \beta_n f_2(x) + p_n(x)$$

and

$$\hat{f}_1(L_n(x)) = \hat{\alpha}_n \hat{f}_1(x) + \hat{\beta}_n \hat{f}_2(x) + \hat{p}_n(x),$$

it is observed that, for $f_1, \hat{f}_1 \in \mathbb{V}_0$,

$$\langle f_1, \hat{f}_1 \rangle = \frac{\sum_{n=1}^N a_n \left(\alpha_n \hat{\beta}_n \langle f_1, \hat{f}_2 \rangle + \beta_n \hat{\alpha}_n \langle f_2, \hat{f}_1 \rangle + \beta_n \hat{\beta}_n \langle f_2, \hat{f}_2 \rangle \right. \\ \left. + \alpha_n \langle f_1, \hat{p}_n \rangle + \hat{\alpha}_n \langle \hat{f}_1, p_n \rangle + \beta_n \langle f_2, \hat{p}_n \rangle \right. \\ \left. + \hat{\beta}_n \langle \hat{f}_2, p_n \rangle + \langle p_n, \hat{p}_n \rangle \right)}{1 - \sum_{n=1}^N a_n \alpha_n \hat{\alpha}_n} \quad (3.3)$$

where, a_n , α_n and β_n , p_n ; $\hat{\alpha}_n$ and $\hat{\beta}_n$, \hat{p}_n , are given by (2.1), (2.3), (2.5) respectively for the interpolation data $\{(x_i, y_i, z_i) : i = 0, 1, \dots, N\}$ and $\{(x_i, \hat{y}_i, \hat{z}_i) : i = 0, 1, \dots, N\}$. Using (3.3), the set of orthogonal functions that forms the Riesz basis of set \mathbb{V}_0 is constructed as follows:

Let, the free variables α_j , γ_j and constrained variables β_j , $j = 1, \dots, N$, $N > 1$, in the construction of CHFIF be chosen such that $\alpha_j + \beta_j \neq \gamma_j$ for atleast one j . Consider, the points y_i and $z_i \in R^{n+1}$, $i = 0, \dots, N$, given by

$$\left. \begin{aligned} y_0 &= (1, r_1, \dots, r_{N-1}, 0), & y_N &= (0, s_1, \dots, s_{N-1}, 1), \\ y_i &= (0, \dots, 1, \dots, 0), & i &= 1, \dots, N-1, \\ y_{N+1+i} &= (0, u_{i,1}, \dots, u_{i,N-1}, 0), & i &= 0, \dots, N; \end{aligned} \right\} \quad (3.4)$$

$$z_i = (0, \dots, 0), \quad z_{N+1+i} = (0, \dots, 1, \dots, 0), \quad i = 0, \dots, N \quad (3.5)$$

and a set of $2(N+1)$ functions $\tilde{f}_i = (\tilde{f}_{i,1}, \tilde{f}_{i,2}) \in \mathcal{S}_0$, $i = 0, \dots, 2N+1$, where the CHFIF $\tilde{f}_{i,1}$ passes through the points (x_k, y_{i_k}) , $k = 0, \dots, N+1$, y_{i_k} being the k^{th} component of y_i and AFIF $\tilde{f}_{i,2}$ passes through the points (x_k, z_{i_k}) , $k = 0, \dots, N+1$, z_{i_k} being the k^{th} component of z_i . Let the function $\tilde{f}_i^* : \mathbb{R} \rightarrow \mathbb{R}^2$, $i = 0, 1, \dots, 2N+1$, be the extension of the function $\tilde{f}_i : I \rightarrow \mathbb{R}^2$ such that $\tilde{f}_i^*(x) = \tilde{f}_i(x)$ for $x \in I$ and $\tilde{f}_i^*(x) = (0, 0)$ for $x \notin I$.

For ensuring the orthogonality of the functions $\tilde{f}_{i,1}^*$ with respect to the inner product in $L_2(\mathbb{R})$, let the values of r_i , s_i and $u_{i,j}$, $i, j = 1, \dots, N-1$, in (3.4) be chosen such that

$$\langle \tilde{f}_{i,1}, \tilde{f}_{0,1} \rangle = 0, \quad \langle \tilde{f}_{i,1}, \tilde{f}_{N,1} \rangle = 0, \quad \langle \tilde{f}_{N+1+j,1}, \tilde{f}_{i,1} \rangle = 0. \quad (3.6)$$

Let, for $i = 1, 2, \dots, N-1$,

$$\zeta_i = \langle \tilde{f}_{N+1+i,1}, \tilde{f}_{0,1} \rangle \quad \text{and} \quad \eta_i = \langle \tilde{f}_{N+1+i,1}, \tilde{f}_{N,1} \rangle. \quad (3.7)$$

The free variables α_j, γ_j and constrained variables $\beta_j, j = 1, 2, \dots, N$, in (2.3) are $3N$ variables and $\zeta_i = \eta_i = 0, i = 1, \dots, N - 1$ is a system of $2N - 2$ equations. Suppose there exist no α_j, γ_j and $\beta_j, j = 1, \dots, N$, in $(-1, 1)$ such that $\zeta_i = \eta_i = 0, i = 1, \dots, N - 1$, then dimension of $S_0^1 < 2N$, which is a contradiction. Hence, there exists atleast one set of α_j, γ_j and $\beta_j, j = 1, \dots, N$, in $(-1, 1)$ such that $\zeta_i = \eta_i = 0, i = 1, \dots, N - 1$. The free variables α_j, γ_j and constrained variables $\beta_j, j = 1, 2, \dots, N$, in (2.3) are chosen such that, for $i = 1, 2, \dots, N - 1, \zeta_i = 0$ and $\eta_i = 0$.

It is easily seen that the functions $\tilde{f}_i^*, i = 0, \dots, 2N + 1, \tilde{f}_{i,1}^*, i = 0, \dots, N$ and the functions $\tilde{f}_{j,2}^*, j = N + 1, \dots, 2N + 1$, are linearly independent. Now, by (2.8), $\tilde{f}_{j,1}^*(x) = \alpha_n \tilde{f}_{j,1}^*(L_n^{-1}(x)) + \beta_n \tilde{f}_{j,2}^*(L_n^{-1}(x)) + p_{n,j}(L_n^{-1}(x))$ and $\tilde{f}_{j,2}^*(x) = \gamma_n \tilde{f}_{j,2}^*(L_n^{-1}(x)) + q_{n,j}(L_n^{-1}(x)), j = 0, 1, \dots, 2N + 1$, where $p_{n,j}$ and $q_{n,j}$ are linear polynomials. By (3.5), the functions $\tilde{f}_{j,2}^*, j = N + 2, \dots, 2N$, are not linear polynomials. Hence, $\sum_{k=1}^{N-1} a_k \tilde{f}_{N+1+k,1}^*(x) - \alpha_n \sum_{k=1}^{N-1} a_k \tilde{f}_{N+1+k,1}^*(L_n^{-1}(x)) = 0$ if and only if $a_k = 0$, which implies $\tilde{f}_{N+1+k,1}^*, k = 1, \dots, N - 1$, are linearly independent. The linear independence of $\tilde{f}_{k,1}^*, \tilde{f}_{N+1+k,1}^*, k = 1, \dots, N - 1$ together with (3.6) will yield same number of orthogonal functions in Gram-Schmidt process.

Let $\{\phi_{i,1}\}_{i=1}^{2N-1} \subset \mathbb{V}_0, i \neq N$, be a sequence of orthogonal functions obtained from $\{\tilde{f}_{i,1}^*\}_{i=1}^{2N}, i \neq N, N + 1$ by the Gram-Schmidt process. Set,

$$\phi_{N,1} = \begin{cases} \tilde{f}_{N,1}^*(x) & x \in [0, 1) \\ \tilde{f}_{0,1}^*(x - 1) & x \in [1, 2) \\ 0 & \text{otherwise} \end{cases}$$

It is easily seen by Proposition 2.2 that $\phi_{i,1}, i = 1, 2, \dots, 2N - 1$, are non-zero functions. Further, by (3.6) and (3.7), it follows that $\{\phi_{i,1} : i = 1, 2, \dots, 2N - 1\}$ is an orthogonal set. This is the set required for the generation of multiresolution analysis of $L_2(\mathbb{R})$ in the following theorem :

Theorem 3.1. *Let free variables α_j, γ_j and constrained variables $\beta_j, j = 1, \dots, N, N > 1$, in the construction of CHFIF be chosen such that $\alpha_j + \beta_j \neq \gamma_j$ for atleast one j and ζ_i, η_i given by (3.7) be such that $\zeta_i = 0, \eta_i = 0, i = 1, \dots, N - 1$. Then,*

$$\mathbb{V}_0 = \text{clos}_{L^2} \text{span}\{\phi_{i,1}(\cdot - l) : i = 1, \dots, 2N - 1, l \in \mathbb{Z}\}, \quad (3.8)$$

where, $\phi_{i,1} \in \mathbb{V}_0$. Also, the set $\{\phi_{i,1}\}_{i=1}^{2N-1}$ generates a continuous, compactly supported multiresolution analysis of $L_2(\mathbb{R})$.

Proof. It is obvious that functions $\phi_{i,1}, i = 1, \dots, 2N-1$, are compactly supported and are elements of \mathbb{V}_0 . Now, $f \in \mathbb{V}_0$ implies $f|_{[i-1,i)} = g_1|_{[i-1,i)}$ is CHFIF for some $g = (g_1, g_2) \in \tilde{\mathbb{V}}_0$. Since, every $g = (g_1, g_2) \in \tilde{\mathbb{V}}_0$ is determined by $g_1(\frac{i}{N})$ and $g_2(\frac{i}{N})$, $i \in \mathbb{Z}$, the function g has unique expansion in terms of the functions $\tilde{f}_i^* = (\tilde{f}_{i,1}^*, \tilde{f}_{i,2}^*)$, $i = 0, \dots, 2N+1$, and their integer translates. Hence, the function $f = g_1 \in \mathbb{V}_0$ has a unique expansion in terms of the functions $\tilde{f}_{i,1}^*$, $i = 1, \dots, N-1, N+2, \dots, 2N-2$, $\phi_{N,1}$ and their integer translates. Thus, CHFIF f has a unique expansion in terms of the functions $\phi_{i,1}$, $i = 1, \dots, 2N-1$, and their integer translates, i.e. $f = \sum_k \left(\sum_{i=1}^{2N-1} K_{k,i} \phi_{i,1}(x-k) \right)$ where, $K_{k,i} = \int_{\mathbb{R}} f(x) \phi_{i,1}(x-k) dx$. Since $f \in \mathbb{V}_0$ is arbitrary, $\mathbb{V}_0 = \text{span}\{\phi_{i,1}(\cdot - l), i = 1, \dots, 2N-1, l \in \mathbb{Z}\}$. Let $\{f_{n,1}\}$ be a sequence in \mathbb{V}_0 such that $\lim_{n \rightarrow \infty} f_{n,1} = f_1^*$ and $\{f_n = (f_{n,1}, f_{n,2})\}$ be a convergent sequence in \mathbb{V}_0 , where $f_{n,2}$ are AFIFs. Since $\tilde{\mathbb{V}}_0$ is closed, $\lim_{n \rightarrow \infty} f_n = f^* = (f_1^*, f_2^*) \in \tilde{\mathbb{V}}_0$ which gives $f_1^*|_{[i-1,i)}, i \in \mathbb{Z}$ is a CHFIF. Hence, $f_1^* \in \mathbb{V}_0$. Therefore, \mathbb{V}_0 is closed and $\mathbb{V}_0 = \text{clos}_{L^2} \text{span}\{\phi_{i,1}(\cdot - l), i = 1, \dots, 2N-1, l \in \mathbb{Z}\}$.

Now, it is shown that the set $\{\phi_{i,1}\}_{i=1}^{2N-1}$ generates a continuous, compactly supported multiresolution analysis of $L_2(\mathbb{R})$:

(a) By Proposition 3.1, it follows that $\dots \supseteq \mathbb{V}_{-1} \supseteq \mathbb{V}_0 \supseteq \mathbb{V}_1 \supseteq \dots$

(b) To prove that $\bigcap_{k \in \mathbb{Z}} \mathbb{V}_k = \{0\}$, let $I_n = [n, n+1], n \in \mathbb{Z}$, and $U_0 = \{f_{\chi_{I_0}} : f \in \mathbb{V}_0\}$ where, $f_{\chi_{I_0}}(x) = f(x)$ if $x \in I_0$ and $f_{\chi_{I_0}}(x) = 0$ if $x \notin I_0$. Since the space U_0 is finite dimensional over \mathbb{R} , the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{L^2}$ restricted to U_0 are equivalent. Hence, there exist a positive constant c such that $\|f\|_\infty \leq c\|f\|_{L^2}$ for all $f \in U_0$. By the property of translation invariance, it is observed that $\|f_{\chi_{I_n}}\|_\infty \leq c\|f_{\chi_{I_n}}\|_{L^2}$ for any $f \in U_0$. Thus, $\|f\|_\infty \leq \sup_n \|f_{\chi_{I_n}}\|_\infty \leq c \sum_{n \in \mathbb{Z}} \|f_{\chi_{I_n}}\|_{L^2} = c\|f\|_{L^2}$ for any $f \in \mathbb{V}_0$. It therefore follows by the definition of \mathbb{V}_k that $\|f\|_\infty \leq cN^{k/2}\|f\|_{L^2}$ for all $f \in \mathbb{V}_k$. Consequently, if $f \in \bigcap_{k \in \mathbb{Z}} \mathbb{V}_k$, then $\|f\|_\infty = 0$ which implies $f = 0$.

(c) For showing that $\text{clos}_{L^2} \bigcup_{m \in \mathbb{Z}} \mathbb{V}_m = L^2(\mathbb{R})$, let $f = (f_1, f_2) \in \tilde{\mathbb{V}}_0$, where f_1 is a CHFIF passing through $(x_k, 1)$ and f_2 is an AFIF passing through (x_k, z_k) . For all $x \in \mathbb{R}$, by (3.4),

$$f_1 = \sum_k \left(\sum_{i=1}^{N-1} C_{k,i} \tilde{f}_{i,1}^*(x-k) + \phi_N(x-k) + \sum_{i=1}^{N-1} D_{k,i} \tilde{f}_{N+1+i,1}^*(x-k) \right)$$

where,

$$C_{k,i} = \left(1 - r_i - s_i - \sum_{j=1}^{N-1} u_{j,i} z_j \right) \quad \text{and} \quad D_{k,i} = z_i, \quad i = 1, \dots, N-1.$$

Now, since $\phi_{i,1}$ are continuous and compactly supported, by using (b) and Proposition 3.1 of [5], it follows that $\bigcup_{k \in \mathbb{Z}} \mathbb{V}_k$ is dense in $L_2(\mathbb{R})$.

(d) For proving that the functions $\phi_{i,1}$, $i = 1, \dots, 2N - 1$, and their integer translates form a Riesz basis for \mathbb{V}_0 , let τ be the smallest eigenvalue of the matrix

$$\frac{1}{\|\phi_{N,1}\|^2} \begin{pmatrix} (\int_I |f_{0,1}(x)|^2 dx)^{1/2} & (\int_I |f_{0,1}(x)| |f_{N,1}(x)| dx)^{1/2} \\ (\int_I |f_{0,1}(x)| |f_{N,1}(x)| dx)^{1/2} & (\int_I |f_{N,1}(x)|^2 dx)^{1/2} \end{pmatrix}.$$

Since $f_{0,1}$ and $f_{N,1}$ are linearly independent, the determinant of the matrix is positive which implies $\tau > 0$. Taking $A = \sqrt{\tau} \|\phi_{N,1}\|_{L_2}$ and $B = \sqrt{3} \|\phi_{N,1}\|_{L_2}$, it is seen that, for every $c = \{c_i\} \in l^2$, $A\|c\|_{l^2} \leq \|\sum c_i \phi_{N,1}(\cdot - i)\|_{L^2} \leq B\|c\|_{l^2}$.

Further, the functions $\phi_{i,1}$, $i = 1, \dots, 2N - 1, i \neq N$, and their integer translates are mutually orthogonal. Therefore, the functions $\phi_{i,1}$, $i = 1, \dots, 2N - 1$, and their integer translates form a Riesz basis for \mathbb{V}_0 . \square

Remark 3.2. By Theorem 3.1, it follows that the set $\{\hat{\phi}_{i,1} : i = 1, 2, \dots, 2N - 1\} \subset \mathbb{V}_0$, where $\hat{\phi}_{i,1}(x) = \phi_{i,1}(x)/\|\phi_{i,1}\|_{L^2}$, $i = 1, 2, \dots, 2N - 1$, generates a continuous, compactly supported multiresolution analysis of $L_2(\mathbb{R})$ by orthonormal functions.

4 Construction of Orthogonal Scaling Functions and Wavelets

For generating multiresolution analysis in Theorem 3.1, the need of orthogonality of functions $\phi_{i,1} \in \mathbb{V}_0$ (c.f. (3.1)), $i = 1, 2, \dots, 2N - 1$, requires that CHFIFs $\tilde{f}_{i,1}$ satisfy conditions (3.6) and (3.7). In this section, as an example, the values of parameters in (3.4) are determined so that the CHFIFs $\tilde{f}_{i,1}$ satisfy these conditions. For simplicity in our presentation, such an example is given for $N = 2$ or, equivalently, when the dimension of vector space \mathcal{S}_0^1 is 4. In addition to requiring that (3.6) and (3.7) hold for functions $\tilde{f}_{i,1}$, $i = 0, 1, 2, 3$, a condition is derived so that the CHFIFs $\tilde{f}_{0,1}$ and $\tilde{f}_{2,1}$ are also orthogonal. The orthogonality of the later CHFIFs implies the orthogonality of $\phi_{1,1}$ with its integer translates that is needed for the construction of orthogonal wavelets. Finally, using these explicitly constructed orthogonal functions $\phi_{i,1}$, $i = 1, 2, 3$, the orthogonal wavelets in the space $\mathbb{V}_{-1} \setminus \mathbb{V}_0$ (c.f. (3.2)) are constructed in this section for $N = 2$.

To find the values of parameters in (3.4), consider CHFIFs $\tilde{f}_{i,1}$, $i = 0, 1, 2, 3$, as follows:

- (I) $\tilde{f}_{0,1}$ corresponds to the data $((0, 1), (\frac{1}{2}, r_1), (1, 0))$,
- (II) $\tilde{f}_{1,1}$ corresponds to the data $((0, 0), (\frac{1}{2}, 1), (1, 0))$,
- (III) $\tilde{f}_{2,1}$ corresponds to the data $((0, 0), (\frac{1}{2}, s_1), (1, 1))$,
- (IV) $\tilde{f}_{3,1}$ corresponds to the data $((0, 0), (\frac{1}{2}, u_{1,1}), (1, 0))$,
where r_1 , s_1 and $u_{1,1}$ are real parameters given by (3.4).

Further, AFIF $\tilde{f}_{i,2}$, $i = 0, 1, 2, 3$, are chosen such that

- (V) $\tilde{f}_{i,2}$, $i = 0, 1, 2$ corresponds to the data $((0, 0), (\frac{1}{2}, 0), (1, 0))$,
- (VI) $\tilde{f}_{3,2}$ corresponds to the data $((0, 0), (\frac{1}{2}, 1), (1, 0))$.

It is observed that $\tilde{f}_{i,1}$, $i = 0, 1, 2$ are self-affine as $\tilde{f}_{i,2}$, $i = 0, 1, 2$ are zero functions. By (2.4), the values of $t_i(x)$, $i = 1, 2$, given by (2.7), in the construction of CHFIF $\tilde{f}_{j,1}$, $j = 0, 1, 2, 3$, are found as the following:

- (i) for CHFIF $\tilde{f}_{0,1}$, $t_1(x) = ((r_1 + \alpha_1 - 1)x + (1 - \alpha_1), 0)$ and $t_2(x) = ((\alpha_2 - r_1)x + (r_1 - \alpha_2), 0)$
- (ii) for CHFIF $\tilde{f}_{1,1}$, $t_1(x) = (x, 0)$ and $t_2(x) = (1 - x, 0)$
- (iii) for CHFIF $\tilde{f}_{2,1}$, $t_1(x) = ((s_1 - \alpha_1)x, 0)$ and $t_2(x) = ((1 - s_1 - \alpha_2)x + s_1, 0)$ and
- (iv) for CHFIF $\tilde{f}_{3,1}$, $t_1(x) = (u_{1,1}x, x)$ and $t_2(x) = ((1 - x)u_{1,1}, 1 - x)$.

The inner products given by (3.6), (3.7) and $\langle \tilde{f}_{0,1}, \tilde{f}_{2,1} \rangle$ are computed using (3.3), wherein the values of $t_i(x)$, $i = 1, 2$, given by (i)-(iv) above, are used. Using (3.6), the values of real parameters r_1 and s_1 given in (3.4) are computationally obtained as

$$r_1 = \frac{4 - 4\alpha_1^2 - 6\alpha_2 - 2\alpha_1\alpha_2 + 3\alpha_1^2\alpha_2 - 4\alpha_2^2 + 3\alpha_2^3}{4(-4 + \alpha_1^2 - \alpha_1\alpha_2 + \alpha_2^2)}, \quad (4.1)$$

and

$$s_1 = \frac{4 - 6\alpha_1 - 4\alpha_1^2 + 3\alpha_1^3 - 2\alpha_1\alpha_2 - 4\alpha_2^2 + 3\alpha_1\alpha_2^2}{4(-4 + \alpha_1^2 - \alpha_1\alpha_2 + \alpha_2^2)}. \quad (4.2)$$

Further, the real parameter $u_{1,1}$, for which (3.6) is satisfied, can be expressed in the form

$$u_{1,1} = \frac{\beta_1 \nu_1(\alpha_1, \alpha_2, \gamma_1, \gamma_2) + \beta_2 \nu_2(\alpha_1, \alpha_2, \gamma_1, \gamma_2)}{2\kappa(\alpha_1, \alpha_2, \gamma_1, \gamma_2)}. \quad (4.3)$$

Finally, the inner products $\zeta = \langle \tilde{f}_{0,1}, \tilde{f}_{3,1} \rangle$ and $\eta = \langle \tilde{f}_{2,1}, \tilde{f}_{3,1} \rangle$ can be expressed as

$$\zeta = \frac{\beta_1 \zeta_1(\alpha_1, \alpha_2, \gamma_1, \gamma_2) + \beta_2 \zeta_2(\alpha_1, \alpha_2, \gamma_1, \gamma_2)}{\kappa(\alpha_1, \alpha_2, \gamma_1, \gamma_2)} \quad (4.4)$$

and

$$\eta = \frac{\beta_1 \eta_1(\alpha_1, \alpha_2, \gamma_1, \gamma_2) + \beta_2 \eta_2(\alpha_1, \alpha_2, \gamma_1, \gamma_2)}{\kappa(\alpha_1, \alpha_2, \gamma_1, \gamma_2)}. \quad (4.5)$$

Now, using (3.3) and $t_i(x)$, $i = 1, 2$, of CHFIF $\tilde{f}_{j,1}$, $j = 0, 2$, given by (I) and (III) above, it is found that,

$$\langle \tilde{f}_{0,1}, \tilde{f}_{2,1} \rangle = \frac{\left[4(r_1 + s_1)(2 - \alpha_1\alpha_2 - 2\alpha_1^2 - 2\alpha_2^2) + 8r_1s_1(4 + \alpha_1\alpha_2 - \alpha_1^2 - \alpha_2^2) \right. \\ \left. - (\alpha_1^2 + \alpha_2^2)^2 + (1 + \alpha_1^2 + \alpha_2^2)^3 + 4(\alpha_1 + \alpha_2)^2 \right. \\ \left. + \alpha_1^3(2 - 2\alpha_2 + 6r_1) + \alpha_2^3(2 - 2\alpha_1 + 6s_1) \right. \\ \left. - 6r_1\alpha_1(2 - \alpha_2^2) - 6s_1\alpha_2(2 - \alpha_1^2) - 2(\alpha_1\alpha_2 + 3\alpha_1 + 3\alpha_2) \right]}{6(2 - \alpha_1 - \alpha_2)(4 - \alpha_1 - \alpha_2)(2 - \alpha_1^2 - \alpha_2^2)}. \quad (4.6)$$

The substitution of values of r_1 and s_1 from (4.1) and (4.2) in (4.6) gives,

$$\langle \tilde{f}_{0,1}, \tilde{f}_{2,1} \rangle = \frac{\rho(\alpha_1, \alpha_2)}{12(-4 + \alpha_1^2 - \alpha_1\alpha_2 + \alpha_2^2)(8 - 6\alpha_1 + \alpha_1^2 - 6\alpha_2 + 2\alpha_1\alpha_2 + \alpha_2^2)} \quad (4.7)$$

where,

$$\rho(\alpha_1, \alpha_2) = 8 + 12\alpha_1 - 28\alpha_1^2 + 6\alpha_1^3 + 2\alpha_1^4 + 12\alpha_2 - 14\alpha_1\alpha_2 + 18\alpha_1^2\alpha_2 - 7\alpha_1^3\alpha_2 \\ - 28\alpha_2^2 + 18\alpha_1\alpha_2^2 + 6\alpha_2^3 - 7\alpha_1\alpha_2^3 + 2\alpha_2^4. \quad (4.8)$$

Thus, by (4.8), $\langle \tilde{f}_{0,1}, \tilde{f}_{2,1} \rangle = 0$ if and only if $\rho(\alpha_1, \alpha_2) = 0$.

To find the orthogonal scaling functions from CHFIF, the free variables $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ and constrained variables β_1, β_2 need to be chosen such that ζ, η given by (4.4), (4.5) respectively satisfy $\zeta = 0$, $\eta = 0$ and $\rho(\alpha_1, \alpha_2) = 0$. It is found that, with $\alpha_1 = 0$, $\alpha_2 = -3 + \sqrt{7}$, $\gamma_1 = \frac{-9}{10}$ and $\gamma_2 = \frac{1}{10}(-67 + 29\sqrt{7})$,

$$\zeta_1(\alpha_1, \alpha_2, \gamma_1, \gamma_2)\eta_2(\alpha_1, \alpha_2, \gamma_1, \gamma_2) - \eta_1(\alpha_1, \alpha_2, \gamma_1, \gamma_2)\zeta_2(\alpha_1, \alpha_2, \gamma_1, \gamma_2) = 0 \quad (4.9)$$

and $\rho(\alpha_1, \alpha_2) = 0$. Writing (4.4) and (4.5) in matrix form and using (4.9), it is observed that, there exist infinitely many values of β_1 and β_2 satisfying $\zeta = 0$, $\eta = 0$ and $\rho(\alpha_1, \alpha_2) = 0$ with the above values of α_1 , α_2 , γ_1 and γ_2 . For example, one pair of such values of β_1

and β_2 is given by $\beta_1 = \frac{1}{20}$ and $\beta_2 = \frac{1}{20}(3 - \sqrt{7})$. Consequently, the following theorem is obtained from the above analysis:

Theorem 4.1. *The functions $\tilde{f}_{i,1}$, $i = 0, 1, 2, 3$, given by (I)-(IV) above, are orthogonal CHFIFs in \mathcal{S}_0^1 if and only if the following conditions are satisfied:*

- (a) $|\alpha_1| < 1, |\alpha_2| < 1, |\gamma_1| < 1, |\gamma_2| < 1, |\beta_1| + |\gamma_1| < 1$ and $|\beta_2| + |\gamma_2| < 1$
- (b) The values of r_1, s_1 and $u_{1,1}$ are given by (4.1), (4.2) and (4.3), respectively.
- (c) The function $\rho(\alpha_1, \alpha_2)$, given by (4.8), is such that $\rho(\alpha_1, \alpha_2) = 0$.
- (d) The functions ζ and η , given by (4.4) and (4.5), respectively are such that $\zeta = 0$ and $\eta = 0$.

Proof. (a) The functions $\tilde{f}_{i,1}$, $i = 0, 1, 2, 3$, given by (I)-(IV), are CHFIFs if and only if the identity (a) holds.

(b) It follows by the choice of functions as in (i)-(iv) that the values of r_1, s_1 and $u_{1,1}$ are given by (4.1), (4.2) and (4.3) if and only if $\langle \tilde{f}_{0,1}, \tilde{f}_{1,1} \rangle = 0$, $\langle \tilde{f}_{1,1}, \tilde{f}_{2,1} \rangle = 0$ and $\langle \tilde{f}_{1,1}, \tilde{f}_{3,1} \rangle = 0$ respectively.

(c) By (4.7), it follows that, $\rho(\alpha_0, \alpha_1) = 0$ if and only if $\langle \tilde{f}_{0,1}, \tilde{f}_{2,1} \rangle = 0$.

(d) By (4.4) and (4.5), it is seen that, $\zeta = 0$ and $\eta = 0$ if and only if $\langle \tilde{f}_{0,1}, \tilde{f}_{3,1} \rangle = 0$ and $\langle \tilde{f}_{2,1}, \tilde{f}_{3,1} \rangle = 0$ respectively.

Thus, the functions $\tilde{f}_{i,1}$, $i = 0, 1, 2, 3$, given by (I)-(IV) above, are orthogonal functions in \mathcal{S}_0^1 if and only if conditions (a)-(d) are satisfied. \square

Having constructed orthogonal functions $\tilde{f}_{i,1} \in \mathcal{S}_0^1$, $i = 0, 1, 2, 3$, that lead to generation of a multiresolution analysis of $L_2(\mathbb{R})$ for $N = 2$, by explicitly determined orthogonal functions $\phi_{i,1}$, $i = 1, 2, 3$, it is natural to seek how orthogonal compactly supported continuous wavelets based on the functions $\phi_{i,1}$, are constructed in this case. For developing such wavelets based on the vector $\Phi_1 = [\phi_{1,1}, \phi_{2,1}, \phi_{3,1}]^t$ of orthogonal scaling functions and their integer translates, define the wavelet spaces \mathbb{W}_k , $k \in \mathbb{Z}$, as the orthogonal complement of \mathbb{V}_k (c.f. (3.2)) in \mathbb{V}_{k-1} . Consider the vector $\Psi_1 = [\psi_{1,1}, \psi_{2,1}, \psi_{3,1}]^t$, where $\psi_{i,1} \in \mathbb{W}_0$, supported on $[0, 2]$, are CHFIFs orthogonal to $\phi_{i,1}$, $i = 1, 2, 3$, and their integer translates. By (3.2), the functions $\psi_{i,1} \in \mathbb{V}_{k-1}$, $i = 1, 2, 3$, are first components of functions $\psi_i = (\psi_{i,1}, \psi_{i,2}) \in \tilde{\mathbb{V}}_{k-1}$, where $\psi_{i,2}$, $i = 1, 2, 3$, are AFIFs supported on $[0, 2]$. The set $\{\psi_{1,1}, \psi_{2,1}, \psi_{3,1}\}$ is called a ‘set of wavelets’ associated with the scaling functions $\phi_{1,1}, \phi_{2,1}, \phi_{3,1}$ if $\{\psi_{i,1}(\cdot - l), i = 1, 2, 3, l \in \mathbb{Z}\}$ forms a

Riesz basis of the space \mathbb{W}_0 . The following theorem gives the existence and construction of non-zero functions $\psi_{i,1}$, $i = 1, 2, 3$, orthogonal to $\phi_{i,1}$, $i = 1, 2, 3$, and their integer translates:

Theorem 4.2. *Let $\phi_{i,1} \in \mathbb{W}_0$, $i = 1, 2, 3$, be orthogonal scaling functions satisfying (3.8). Then, there exist non-zero functions $\psi_{i,1} \in \mathbb{W}_0$, orthogonal to $\phi_{i,1}$, $i = 1, 2, 3$, and their integer translates.*

Proof. Set

$$\psi_{i,1} = \begin{cases} g_{1,i,1}(x) & \text{for } 0 \leq x \leq \frac{1}{2} \\ g_{1,i,2}(x) & \text{for } \frac{1}{2} \leq x \leq 1 \\ g_{1,i,3}(x) & \text{for } 1 \leq x \leq \frac{3}{2} \\ g_{1,i,4}(x) & \text{for } \frac{3}{2} \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \quad i = 1, 2, 3 \quad (4.10)$$

and

$$\psi_{i,2} = \begin{cases} g_{2,i,1}(x) & \text{for } 0 \leq x \leq \frac{1}{2} \\ g_{2,i,2}(x) & \text{for } \frac{1}{2} \leq x \leq 1 \\ g_{2,i,3}(x) & \text{for } 1 \leq x \leq \frac{3}{2} \\ g_{2,i,4}(x) & \text{for } \frac{3}{2} \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \quad i = 1, 2, 3 \quad (4.11)$$

where, CHFIFs $g_{1,i,j}$, $i = 1, 2, 3$, $j = 1, 2, 3, 4$, corresponds to the data $\{(x_k, A_{i,l}) : l = 2j + k - 2, k = 0, 1, 2\}$, $A_{i,0} = 0 = A_{i,8}$ and AFIFs $g_{2,i,j}$, $i = 1, 2, 3$, $j = 1, 2, 3, 4$, corresponds to the data $\{(x_k, B_{i,l}) : l = 2j + k - 2, k = 0, 1, 2\}$, $B_{i,0} = 0 = B_{i,8}$. It is observed that there are 42 unknowns values, $A_{i,l}$ and $B_{i,l}$, $i = 1, 2, 3$ and $l = 1, \dots, 7$.

The values of $A_{i,l}$ and $B_{i,l}$, $i = 1, 2, 3$, $l = 1, \dots, 7$, are found such that

- (A) $\psi_{i,1}$ is orthogonal to $\phi_{j,1}$ and their integer translates for $i, j = 1, 2$ & 3,
- (B) $\psi_{i,1}$ is orthogonal to $\psi_{j,1}$ for $i \neq j$, $i, j = 1, 2$ & 3,
- (C) $\psi_{i,2}$ is orthogonal to $\phi_{2,2}$ for $i = 1, 2$ & 3 and their integer translates and
- (D) $\psi_{i,2}$ is orthogonal to $\phi_{j,1}$ for $i = 1, 2$ & 3, $j = 1, 2$.

These conditions (A)-(D) give rise to a set of 36 non-linear equations. The functions $\psi_{i,1}$ for $i = 1, 2$ & 3 are normalized to give 3 more equations. Thus, there are 39 non-linear equations with 42 variables. Hence, there are 3 free variables. Solving these equations, we get the values of $A_{i,l}$ and $B_{i,l}$, $i = 1, 2, 3$, $l = 1, \dots, 7$.

Now to show there exist such non-zero functions, let $\alpha_1 = 0$, $\alpha_2 = -3 + \sqrt{7}$, $\gamma_1 = \frac{-9}{10}$, $\gamma_2 = \frac{1}{10}(-67 + 29\sqrt{7})$, $\beta_1 = \frac{1}{20}$ and $\beta_2 = \frac{1}{20}(3 - \sqrt{7})$. Then, the values of r_1, s_1 and $u_{1,1}$ (c.f. (4.1), (4.2) and (4.3)) for ensuring the orthogonality of $\phi_{i,1}$, $i = 1, 2, 3$, are obtained as $r_1 = -3 + \sqrt{7}$, $s_1 = \frac{1}{6}(-4 + \sqrt{7})$ and $u_{1,1} = \frac{-371-40\sqrt{7}}{70245}$. Further, it is observed that $\zeta_1 = 0$ and $\eta_1 = 0$ (c.f. (4.4) and (4.5)). Therefore, $\phi_{i,1} \in \mathbb{W}_0$, $i = 1, 2, 3$, are orthogonal scaling functions satisfying (3.8).

One possible values of $A_{i,l}$ and $B_{i,l}$, $i = 1, 2, 3$, and $l = 1, \dots, 7$, is obtained as

$$A_{1,1} = -1.04784, A_{1,2} = 0.0125935, A_{1,3} = -1.04663, A_{1,4} = 0.0231596,$$

$$A_{1,5} = 0.00599567, A_{1,6} = -0.00795969, A_{1,7} = 0.00391617,$$

$$A_{2,1} = 0, A_{2,2} = 0, A_{2,3} = -0.298716, A_{2,4} = 1.32346, A_{2,5} = -2.4746,$$

$$A_{2,6} = 1.12432, A_{2,7} = -0.553166,$$

$$A_{3,1} = 0, A_{3,2} = 0, A_{3,3} = 0, A_{3,4} = 0,$$

$$A_{3,5} = 1.06312, A_{3,6} = 0, A_{3,7} = 0.983686,$$

$$B_{1,1} = 19.1929, B_{1,2} = -21.6229, B_{1,3} = 11.8901, B_{1,4} = -11.1171,$$

$$B_{1,5} = -4.93066, B_{1,6} = 1.19803, B_{1,7} = 0.567807,$$

$$B_{2,1} = 0, B_{2,2} = 0, B_{2,3} = 0, B_{2,4} = 0,$$

$$B_{2,5} = -11.6825, B_{2,6} = -15.2071, B_{2,7} = -3.19525,$$

$$B_{3,1} = 0, B_{3,2} = 0, B_{3,3} = 0, B_{3,4} = 0,$$

$$B_{3,5} = -13.3015, B_{3,6} = 33.9169, B_{3,7} = -11.0405.$$

Thus, there exist non-zero functions $\psi_{i,1} \in \mathbb{W}_0$, orthogonal to $\phi_{i,1}$, $i = 1, 2, 3$, and their integer translates. \square

The graphs of $\psi_{i,1}$ for $k = 1, 2, 3$ corresponding to the values of $A_{i,l}$ and $B_{i,l}$ given above is shown in the Figure 1.

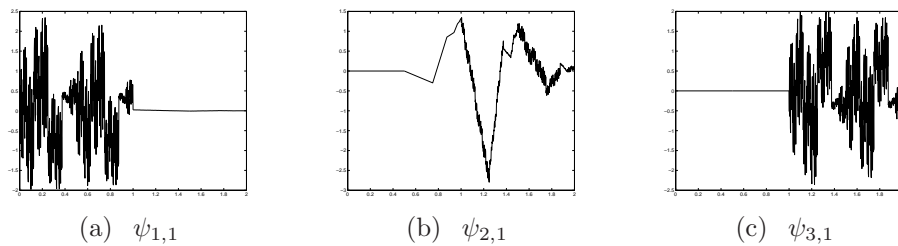


Figure 1: Wavelets $\psi_{i,1}$ for $i = 1, 2$ & 3

In general, if $\phi_{i,1}$, $i = 1, \dots, 2N - 1$, are orthogonal scaling functions satisfying (3.8), then there exist non-zero functions $\psi_{i,1} \in \mathbb{W}_0$, $i = 1, \dots, (2N - 1)(N - 1)$ orthogonal to $\phi_{i,1}$, $i = 1, \dots, 2N - 1$, and their integer translates.

5 Conclusions

In this paper, multiresolution analysis arising from Coalescence Hidden-variable Fractal Interpolation Functions is accomplished. The availability of a larger set of free variables and constrained variables with CHFIF in multiresolution analysis based on CHFIFs provides more control in reconstruction of functions in $L_2(\mathbb{R})$ than that provided by multiresolution analysis based only on affine FIFs. In our approach, the vector space of CHFIFs is introduced, its dimension is determined and Riesz bases of vector subspaces \mathbb{V}_k , $k \in \mathbb{Z}$, consisting of certain CHFIFs in $L_2(\mathbb{R}) \cap C_0(\mathbb{R})$ are constructed. As a special case, for the vector space of CHFIFs of dimension 4, orthogonal bases for the vector subspaces \mathbb{V}_k , $k \in \mathbb{Z}$, are explicitly constructed and, using these bases, compactly supported continuous orthonormal wavelets are generated.

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